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A vortex filament moving without change of form

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The motion of a very thin vortex filament is investigated using the localized induction equation. A family of vortex filaments which move without change of form are obtained. They are expressed in terms of elliptic integrals of the first, second and third kinds. In general they do not close and have infinite lengths. In some particular cases they take the form of closed coils which wind a doughnut. There exist a family of closed vortex filaments which do not travel in space but only rotate around a fixed axis. Our solutions include various well-known shapes such as the circular vortex ring, the helicoidal filament, the plane sinusoidal filament, Euler's elastica and the solitary-wave-type filament. It is shown that they correspond to the travelling wave solution of a nonlinear Schrödinger equation.

1. Introduction

The study of the motion of a vortex filament gives a clue to understanding the properties of the velocity fields at very large Reynolds numbers, such as turbulence and aircraft wakes.

The motion of the vortex filaments in an unbounded perfect fluid is described by the Biot–Savart law. In two-dimensional flow the motion becomes very simple, since the vortex filaments are parallel with each other. Each vortex moves with the velocity induced by the other vortices. There have been many studies of a wide range of problems, from the motion of a vortex pair and trains of vortex filaments (Lamb 1932) to the statistical mechanics of the system of vortices (Onsager 1949).

In the three-dimensional case, on the other hand, a quite different effect appears. Since the vortex filaments have a curvature in general, each vortex is moved by the velocity induced by itself as well as by the others. When the size of core of the vortex is very small compared with its radius of curvature, the motion is governed essentially by the local curvature of the filament (Tung & Ting 1967; Saffman 1970; Fraenkel 1970, 1972; Moore & Saffman 1972). Hama (1962, 1963) derived, on the suggestion of R. J. Arms, the so-called localized induction equation (LIE) (2.1), which is asymptotically valid for the motion of a very thin vortex filament, and used it to investigate the motion of filaments of various shapes numerically. It is interesting to note that this equation is essentially the same as (6.1) which describes a one-dimensional classical spin system (Lakshmanan, Ruijgrok & Thompson 1976; Lakshmanan 1977).

Betchov (1965) derived the intrinsic equations for the curvature and the torsion of the vortex filament from LIE and obtained the helicoidal filaments, which were shown to be unstable to small disturbances.

Starting out from the Frenet–Seret formulae in differential geometry, Hasimoto (1971, 1972) derived the nonlinear Schrödinger equation (NSE) (5.1) for the complex variable whose amplitude is the curvature and whose phase is the torsion of the vortex filament and obtained the plane vortex filaments rotating around a fixed axis and a solitary-wave-type solution which propagates steadily with constant velocity along the vortex filament. This equation, which is equivalent to Betchov's intrinsic equation, is familiar in hydrodynamics, plasma physics, nonlinear optics, solid state physics and so on. It describes self-modulation of a monochromatic wave of various kinds in water waves and plasma waves (Benney & Newell 1967; Karpman & Krushkal 1969; Taniuti & Yajima 1969; Asano, Taniuti & Yajima 1969), two-dimensional self-focusing of a stationary wave beam (Talanov 1965; Kelley 1965), dynamics of a continuum spin system (Lakshmanan 1977), and so on. The properties of NSE and its solution have been extensively investigated. The travelling-wave solutions, which include the nonlinear plane wave and the envelope solitary wave (Saffman 1961) as special cases, were obtained (Asano *et al.* 1969; Zakharov & Shabat 1972, 1973; Hasimoto & Ono 1972; Scott, Chu & McLaughlin 1973). The complete integrability of NSE was shown by Zakharov & Manakov (1974). Recently Lamb (1977) found that the motions of the space curves have an intimate relation with the nonlinear evolution equations which can be solved by inverse scattering methods. The NSE which corresponds to the motion of a vortex filament is of course included as a special case.

In this paper we seek all the shapes of vortex filaments which move steadily with no deformation. We deal with LIE rather than NSE for the following two reasons. Firstly, it is possible in principle but rather cumbersome in practice, except for some simple cases, to determine the shape of the vortex filament from the solution of NSE (see Eisenhart 1960). Secondly, the physical concepts, such as the translational and rotational velocities of the vortex filament, appear explicitly in LIE. The motion of a vortex with no deformation is equivalent to that of a rigid body if we allow its slipping motion along the filament. We derive in § 2 the basic equations which describe the steady motion of a vortex filament. These equations are solved in § 3. The solutions are expressed in terms of elliptic integrals of the first, second and third kinds. They have three parameters and the shapes of the vortex filaments change depending upon them. Several interesting shapes of vortex filaments are examined in § 4. The vortex filaments do not close and have infinite lengths in general. In some particular cases, however, they take the form of closed coils which wind a doughnut. Moreover for some special values of the parameters they reduce to the well-known shapes obtained earlier by other authors, for example, a circular ring, a plane sinusoidal filament (Kelvin 1880), a helicoidal filament (Betchov 1965), Euler's elastica (Hasimoto 1971), the solitary-wave-type filament (Hasimoto 1972) and a coil winding a circular ring (Kambe & Takao 1971). The relation between our solution of LIE and the travelling-wave solution of NSE is discussed in § 5. We note in § 6 the equivalence of the equations between the spin system and the vortex filament.

The stability of the vortex filaments is interesting in connection with that of the solutions of NSE. The stability properties of a vortex filament are known in a few special cases, such as a circular vortex ring and a helicoidal filament (Betchov 1965; Kambe & Takao 1971; Widnall 1972). Correspondingly, the studies of stability of the travelling-wave solutions of NSE are limited within the nonlinear plane wave (Lighthill 1965, 1967; Bespalov & Talanov 1966) and the envelope solitary wave

(Zakharov & Shabat 1972). The study of the stability of the other solutions is now in progress and will be reported in a separate paper.

2. Localized induction equation

The motion of a vortex filament of infinitesimal core in an unbounded perfect fluid is described by the so-called localized induction equation (Hama 1962):

$$\partial \mathbf{x}/\partial t = \mathbf{x}' \times \mathbf{x}'', \tag{2.1}$$

where $\mathbf{x}(s,t)$ denotes the position of the element of the vortex filament, t the time, s the distance measured along the vortex filament and the prime indicates differentiation with respect to s. Note that the interactions with the far distant portions of the filament and with the other filaments, if any, are neglected in this approximation. Since the induced velocity is perpendicular to the vortex filament, it never stretches or contracts. It follows from the definition that

$$|\mathbf{x}'| = 1. \tag{2.2}$$

We consider the motion of a filament whose shape does not change in time. Such a motion is composed of a solid motion and a slipping motion along the filament. A solid motion is generally constructed from a rotational motion and a translational motion. Let us denote the angular velocity of the rotational motion by $\mathbf{\Omega}$ and the velocity of the translational motion by $\mathbf{V} \equiv \mathbf{V}_{\parallel} + \mathbf{V}_{\perp}$, where $\mathbf{V}_{\parallel}$ and $\mathbf{V}_{\perp}$ are the components of $\mathbf{V}$ parallel and perpendicular to $\mathbf{\Omega}$, respectively. Equation (2.1) says that the velocity at an element of the filament is completely determined by the shape and orientation of the filament. We can easily deduce the following conclusions. (i) The angular velocity $\mathbf{\Omega}$ is a constant vector. The instantaneous axis of rotation moves with velocity $\mathbf{V}_{\perp}$. (ii) The parallel component $\mathbf{V}_{\parallel}$ is invariant in time. (iii) The perpendicular component $\mathbf{V}_{\perp}$ rotates in a plane perpendicular to $\mathbf{\Omega}$ with angular velocity $\mathbf{\Omega}$. The magnitude of $\mathbf{V}_{\perp}$ does not change in time. (iv) There is a fixed line for this solid motion, which differs from the instantaneous axis of rotation by $\mathbf{\Omega} \times \mathbf{V}_{\perp}/\Omega^2$, where $\Omega \equiv |\mathbf{\Omega}|$. We choose this fixed line as the z-axis. Then the solid motion is expressed as the sum of the rotational motion around the z-axis with angular velocity $\mathbf{\Omega}$ ($\equiv \Omega\hat{\mathbf{z}}$) and the translational motion parallel to the z-axis with velocity $\mathbf{V}_{\parallel}$ ($\equiv V\hat{\mathbf{z}}$), where $\hat{\mathbf{z}}$ is the unit vector in the z-direction. Let C be the speed of the slipping motion which is constant in time. The velocity at an element of the filament can be written as

$$\partial \mathbf{x}/\partial t = -C\mathbf{x}' + \Omega\hat{\mathbf{z}} \times \mathbf{x} + V\hat{\mathbf{z}}. \tag{2.3}$$

The parameters C, Ω and V are invariant in time.† Then (2.1) becomes

$$-C\mathbf{x}' + \Omega\hat{\mathbf{z}} \times \mathbf{x} + V\hat{\mathbf{z}} = \mathbf{x}' \times \mathbf{x}''. \tag{2.4}$$

The solution of (2.3) is immediately obtained as

$$\mathbf{x}(s,t) = r(\xi)\,\hat{\mathbf{r}}(\xi,t) + \{z(\xi) + Vt\}\,\hat{\mathbf{z}}, \tag{2.5}$$

† The vectors $\mathbf{x}'$, $\hat{\mathbf{z}} \times \mathbf{x}$ and $\hat{\mathbf{z}}$ are linearly independent in general. If they are linearly dependent, the shape of the vortex filament can be expressed by different combinations of C, Ω and V. It can be proved that in such cases the vortex filament takes the form either of a helicoidal curve whose axis is the z-axis or of a straight line parallel to the z-axis.

where

$$\xi = s - Ct, \tag{2.6}$$

and $\hat{\mathbf{r}}$ is a unit vector perpendicular to $\hat{\mathbf{z}}$ and satisfies the equation

$$\partial\hat{\mathbf{r}}/\partial t = \Omega\hat{\mathbf{z}} \times \hat{\mathbf{r}}. \tag{2.7}$$

If we denote the angle between $\hat{\mathbf{r}}$ and a reference direction perpendicular to $\hat{\mathbf{z}}$ by $\Theta(\xi, t)$, then (2.7) gives

$$\Theta(\xi, t) = \theta(\xi) + \Omega t, \tag{2.8}$$

where $\theta(\xi)$ is an arbitrary function of ξ.

The shape of the vortex filament is determined by (2.2) and (2.4), which are solved in the next section. Since $\partial/\partial s = \partial/\partial\xi$, the prime also means differentiation with respect to ξ.

3. Solutions

Since (2.2) and (2.4) have the following symmetries:

$$(C, \Omega, V) \to (C, -\Omega, -V) \quad \text{for} \quad \xi \to -\xi,$$

and

$$(C, \Omega, V) \to (-C, -\Omega, V) \quad \text{for} \quad z \to -z,$$

we can restrict the range of the parameters to

$$\Omega \geqslant 0, \quad C \geqslant 0, \quad -\infty < V < \infty. \tag{3.1}$$

Moreover, since the solutions for $\Omega = 0$ happen to agree with the special cases for $\Omega \neq 0$ (see the appendix), it is sufficient to examine the case $\Omega > 0$. The scale transformation

$$r \to r/\Omega^{\frac{1}{2}}, \quad z \to z/\Omega^{\frac{1}{2}}, \quad \xi \to \xi/\Omega^{\frac{1}{2}}, \quad C \to \Omega^{\frac{1}{2}}C, \quad V \to \Omega^{\frac{1}{2}}V \tag{3.2}$$

reduces the problem to the case $\Omega = 1$.

The inner product of (2.4) with $\Omega = 1$ and $\mathbf{x}'$, together with (2.2), leads to

$$-C + r^2\theta' + Vz' = 0. \tag{3.3}$$

The outer product of (2.4) and $\mathbf{x}'$, on the other hand, gives

$$(\hat{\mathbf{z}} \times \mathbf{x} + V\hat{\mathbf{z}}) \times \mathbf{x}' = \mathbf{x}'', \tag{3.4}$$

the z-component of which is

$$-rr' = z''. \tag{3.5}$$

We have shown that (2.2) and (2.4) imply (3.3) and (3.5). It can be shown that (2.2), (3.3) and (3.5) imply the whole of (3.4), and then that they imply (2.4). Therefore, the system (2.2), (3.3) and (3.5) is equivalent to the system (2.2) and (2.4).

Equation (3.5) is integrated to be

$$z' = \tfrac{1}{2}(A - R), \tag{3.6}$$

where

$$R = r^2, \tag{3.7}$$

and A is a constant of integration. Substituting (3.6) into (3.3), we get

$$\theta' = \tfrac{1}{2}V + (C - \tfrac{1}{2}AV)/R. \tag{3.8}$$

Introduction of (3.6) and (3.8) into (2.2) gives us

$$R'^2+f(R) = 0, \tag{3.9}$$

where

$$f(R) = R^3+(V^2-2A)R^2+(A^2-4-2AV^2+4VC)R+(2C-AV)^2. \tag{3.10}$$

Equation (3.9) represents the motion of a point of unit mass in a potential field $\frac{1}{2}f(R)$. In order that the realizable solutions $R \geqslant 0$ may exist, it is necessary that $f(R) \leqslant 0$ and that the equation

$$f(R) = 0 \tag{3.11}$$

has two non-negative roots and one non-positive root since the constant term in (3.10) is non-negative. We can show that for (3.11) to have such roots it is necessary and sufficient that the parameters A, C and V satisfy the following condition:

$$A \geqslant \frac{2h}{(a^2+h^2)^{\frac{1}{2}}}+a^2, \quad C = \pm\left((a^2+h^2)^{\frac{1}{2}}+\frac{h}{a^2+h^2}\right), \quad V = \pm(h+(a^2+h^2)^{-\frac{1}{2}}), \tag{3.12}$$

where the parameters a and h run over the range $a \geqslant 0$ and $-\infty < h < \infty$, the point $a = h = 0$ being excluded. Note that the boundary surface, which is given by the three equalities in (3.12), corresponds to helicoidal vortex filaments (see (4.6)). The lines of constant A on this surface are plotted in figure 1. We can see that it is symmetric about the A-axis ($C = V = 0$) and that $A \geqslant -2$, where the equality holds on the line $C+V = 0$.

Let the three real roots of (3.11) be $\alpha, \beta, -\gamma$ ($\alpha \geqslant \beta \geqslant 0, \gamma \geqslant 0$). Then the solution of (3.9) is written as

$$\begin{aligned} R &= \alpha+(\beta-\alpha)\,\mathrm{sn}^2(\tfrac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}\xi\,|\,k) \\ &= \beta+(\alpha-\beta)\,\mathrm{cn}^2(\tfrac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}\xi\,|\,k) \\ &= -\gamma+(\alpha+\gamma)\,\mathrm{dn}^2(\tfrac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}\xi\,|\,k), \end{aligned} \tag{3.13}$$

where

$$k = (\alpha-\beta)^{\frac{1}{2}}/(\alpha+\gamma)^{\frac{1}{2}} \tag{3.14}$$

is the modulus of the Jacobian elliptic functions.

Substituting (3.13) into (3.6) and (3.8) and integrating with respect to ξ, we get

$$z = \tfrac{1}{2}(A+\gamma)\xi-(\alpha+\gamma)^{\frac{1}{2}}E(\tfrac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}\xi\,|\,k)+z_0 \tag{3.15}$$

and

$$\theta = \tfrac{1}{2}V\xi+\frac{2C-AV}{\alpha(\alpha+\gamma)^{\frac{1}{2}}}\,\Pi\left(\tfrac{1}{2}(\alpha+\gamma)^{\frac{1}{2}}\xi\,\middle|\,\frac{\alpha-\beta}{\alpha},k\right)+\theta_0, \tag{3.16}$$

where z_0 and θ_0 are constants of integration, and

$$E(u\,|\,k) = \int_0^u \mathrm{dn}^2(u'\,|\,k)\,du' \tag{3.17}$$

and

$$\Pi(u\,|\,l,k) = \int_0^u \frac{du'}{1-l\,\mathrm{sn}^2(u'\,|\,k)} \tag{3.18}$$

are incomplete elliptic integrals of the second and third kinds respectively (Abramowitz & Stegun 1972).

Thus, the shape of the vortex filament in steady motion is expressed by (3.13),

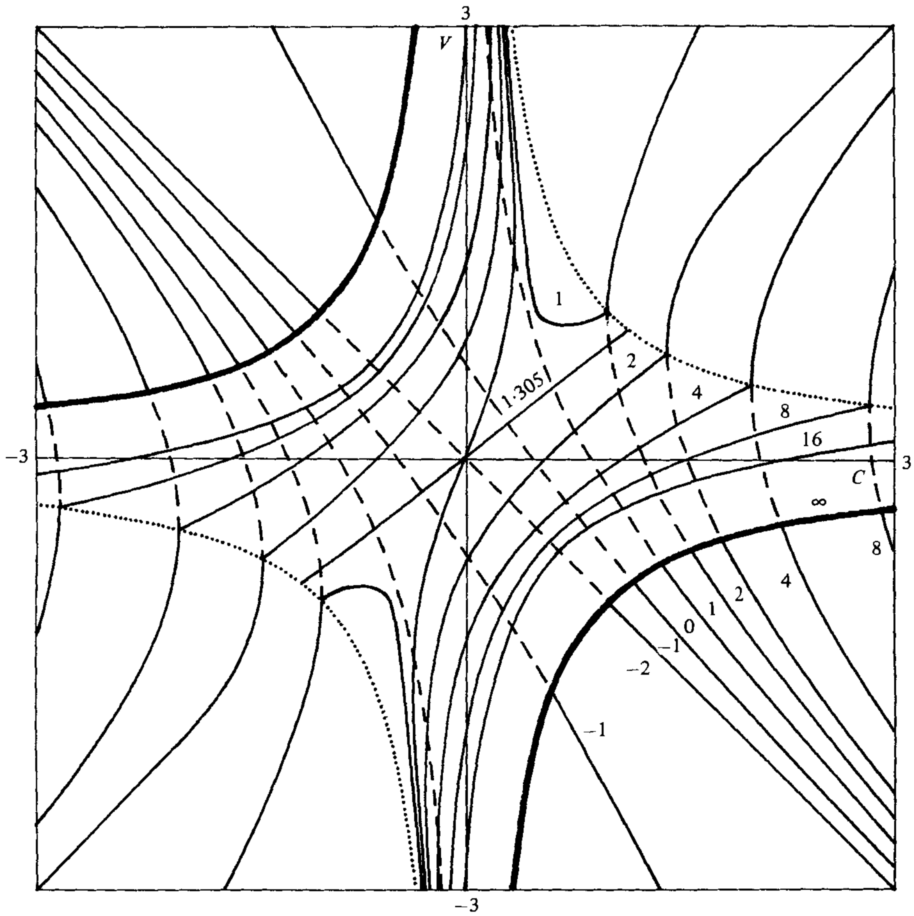

FIGURE 1. The surfaces giving the boundary (4.6) and the periodic condition (4.2). The ordinate and the abscissa are taken to be V and C respectively. The numbers attached to the curves denote the values of A. The dotted and thick lines are $CV = 1$ and $CV = -1$ respectively. The surface (4.2) is limited to be within the area $C^2V^2 \leqslant 1$, while (4.6) extends to infinity. Both the surfaces are symmetric about the A-axis ($C = V = 0$).

(3.15) and (3.16), which take various forms depending upon the three parameters A, C and V or α, β and γ. In general the filament does not close and has infinite length. In the next section the combinations of the parameters which give a closed vortex filament or several other interesting shapes for the filaments are examined.

4. Special cases

The solution obtained in the preceding section is for $\Omega = 1$ (see (3.2)). In order to get the solution for general values of Ω we have only to make the following transformation in the resulting equations:

$$\begin{aligned} &\alpha \to \Omega\alpha, && \beta \to \Omega\beta, && \gamma \to \Omega\gamma, \\ &r \to \Omega^{\frac{1}{2}} r, && z \to \Omega^{\frac{1}{2}} z, && \xi \to \Omega^{\frac{1}{2}}\xi, \\ &A \to \Omega A, && C \to C/\Omega^{\frac{1}{2}}, && V \to V/\Omega^{\frac{1}{2}}. \end{aligned}$$

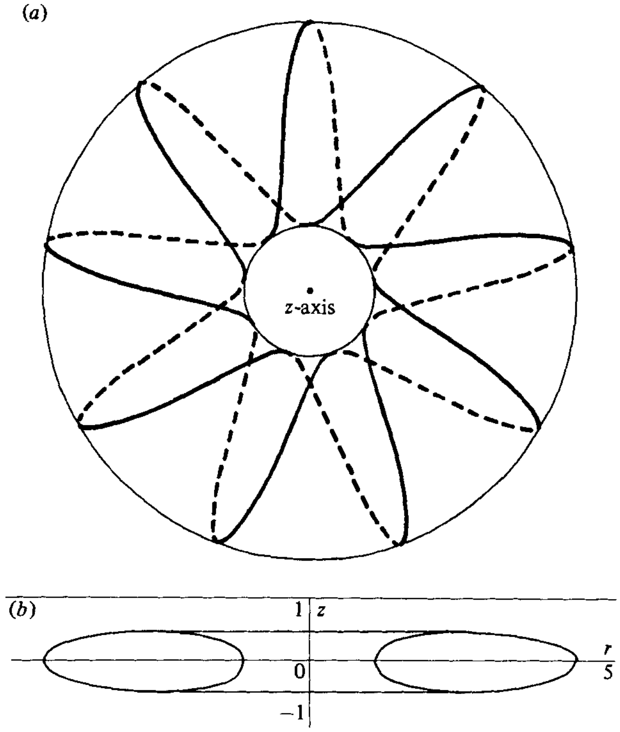

FIGURE 2. An example of closed vortex filaments without translational velocity. (*a*) The projection on the (r, θ) plane. (*b*) The cross-section of the doughnut. Here $m = 9$, $n = 2$, $A = 2{\cdot}486$, $C = 0{\cdot}735$, $\alpha = 4{\cdot}358$, $\beta = 1{\cdot}076$ and $\gamma = 0{\cdot}461$.

4.1. *Closed vortex filament*

Now we begin by considering the condition which gives a closed filaments. The shape of the filament is expressed by (3.13), (3.15) and (3.16). In order that they represent a closed filament, it is necessary that r and z be periodic functions of ξ with the common period. The function r which is expressed by the square root of (3.13) is always periodic and its period is given by

$$4K(k)/(\alpha+\gamma)^{\frac{1}{2}},$$

where $K(k)$ is the complete elliptic integral of the first kind. Since $E(u \mid k)$ has the property that

$$E(u+2K(k) \mid k) = E(u \mid k)+2E(k), \tag{4.1}$$

where $E(k)$ is the complete elliptic integral of the second kind, z is periodic if

$$\frac{A+\gamma}{\alpha+\gamma} = \frac{E(k)}{K(k)}. \tag{4.2}$$

The period of z is the same as that of r. The condition (4.2) defines a surface in the parameter space (A, C, V). The lines of constant A on this surface in the existing region (3.12) are plotted in figure 1. It is connected with the boundary surface (4.6) on the curve $CV = 1$, where the vortex filament becomes a circular ring (see § 4.2). The point $(A, C, V) = (1{\cdot}305, 0, 0)$ is a saddle point. As $A \to \infty$, the equivalue curves of A approach the line $CV = -1$.

If the azimuthal angle θ changes by $2n\pi/m$ over the period, where m and n are integers prime to each other, the filament becomes a closed curve. This condition is written as†

$$\frac{2VK(k)}{(\alpha+\gamma)^{\frac{1}{2}}}+\frac{2(2C-AV)}{\alpha(\alpha+\gamma)^{\frac{1}{2}}}\Pi\left(\frac{\alpha-\beta}{\alpha},k\right)=\frac{2n}{m}\pi, \tag{4.3}$$

where

$$\Pi(l,k)=\Pi(K(k)\,|\,l,k). \tag{4.4}$$

Then, the closed filament winds m times a doughnut, whose cross-section is described by (3.13) and (3.15), and rounds about the z-axis n times. When condition (4.3) is not fulfilled, the vortex filament covers the whole surface of the doughnut.

It is interesting to see that there exists a family of closed vortex filaments without translational velocity ($V=0$). These filaments rotate only around the z-axis. We can show that $m/n>2$ from (4.2) and (4.3) with $V=0$. In the limit $C\to 0$ (and $A\to 1{\cdot}305$), we have $m/n=2$, $\alpha=3{\cdot}305$ and $\beta=0$. Then the hole of the doughnut disappears. As the magnitude of C increases, both the ratio m/n and the size of the hole increase. In the limit $C\to\infty$, we have $m/n\approx 3\surd(3)\,C^2$, $\alpha\approx\beta\approx 3C^2$ and $\alpha-\beta\approx\frac{4}{3}\surd 6$. Then we find a big and thin doughnut. As an example, a filament of $m=9$ and $n=2$ is displayed in figure 2.

4.2. *Helicoidal filaments*

When $\alpha=\beta$ ($\equiv a^2$, say) and $a^2\neq 0$, the solution is a helix in general (for $A\neq a^2$). In the exceptional case in which $A=a^2$ (or $CV=1$ or $h=0$ below), it becomes a circular ring of radius a. If we denote the radius and the pitch of the helix by a and h/a respectively, the solution is written as

$$r=a,\quad \theta=\pm(a^2+h^2)^{-\frac{1}{2}}\xi+\theta_0,\quad z=h(a^2+h^2)^{-\frac{1}{2}}\xi+z_0 \tag{4.5}$$

(Betchov 1965). The parameters are given by

$$A=\frac{2h}{(a^2+h^2)^{\frac{1}{2}}}+a^2,\quad C=\pm\left((a^2+h^2)^{\frac{1}{2}}+\frac{h}{a^2+h^2}\right),\quad V=\pm\left(h+(a^2+h^2)^{-\frac{1}{2}}\right). \tag{4.6}$$

It is seen that each element of the filament moves with speed $-ah/(a^2+h^2)^{\frac{3}{2}}$ and $\pm a^2/(a^2+h^2)^{\frac{3}{2}}$ in the θ- and z-directions respectively. If we imagine that the helicoidal filament is a rigid body, it may be regarded either as rotating around the z-axis with circumferential speed $-a/h(a^2+h^2)^{\frac{1}{2}}$ without translation or as moving along the z-axis with speed $\pm 1/(a^2+h^2)^{\frac{1}{2}}$ without rotation. The fact that we can take such different viewpoints about the motion of the vortex filament arises from the linear dependence of the three vectors on the right-hand side of (2.3). In fact, they are related with each other as

$$\mathbf{x}'=(a^2+h^2)^{-\frac{1}{2}}(\pm\hat{\mathbf{z}}\times\mathbf{x}+h\hat{\mathbf{z}})$$

(see the footnote on p. 399).

4.3. *Double helixes*

When the parameters (A, C, V) deviate slightly from (4.6) with $a\neq 0$, i.e.

$$A=\frac{2h}{(a^2+h^2)^{\frac{1}{2}}}+a^2+\epsilon^2\quad(\epsilon\ll 1), \tag{4.7}$$

and C and V are the same as (4.6), the solution is written as

$$r=a+\frac{\epsilon\cos\mu\xi}{\mu a(a^2+h^2)^{\frac{1}{2}}},\quad z=\frac{h\xi}{(a^2+h^2)^{\frac{1}{2}}}-\frac{\epsilon\sin\mu\xi}{\mu^2(a^2+h^2)^{\frac{1}{2}}}+z_0, \tag{4.8a, b}$$

$$\theta=\pm\left\{\frac{\xi}{(a^2+h^2)^{\frac{1}{2}}}+\left(h-\frac{1}{(a^2+h^2)^{\frac{1}{2}}}\right)\frac{2\epsilon\sin\mu\xi}{\mu^2a^2(a^2+h^2)^{\frac{1}{2}}}\right\}+\theta_0, \tag{4.8c}$$

† We can show, using (4.2), that $m/n\geqslant 1$ in the existing region (3.12).

where

$$\mu = [a^2 + (h - (a^2+h^2)^{-\frac{1}{2}})^2]^{\frac{1}{2}}. \tag{4.9}$$

When $h \neq 0$, this represents a double helix which winds around the helix of (4.5). In the particular case in which $h = 0$ and $a^4 = (m/n)^2 - 1$, we have a closed filament, which takes the form of a coil winding a circular ring of radius a. When $n = 1$, it agrees with the solution obtained in the study of stability of a circular vortex ring by Kambe & Takao (1971).

4.4. *Straight line and small variation thereof*

The most trivial solution is a straight line which is established when $\alpha = \beta = 0$ or $A = \pm 2$, $C = \pm V$ or $a = 0$, which is a special case of the helicoidal filament. Then, the solution coincides with the z-axis, i.e. $r \equiv 0$, and is stationary. Since $\mathbf{x}'$ and $\hat{\mathbf{z}}$ are parallel with each other, $C \pm V$ is indefinite.

In the vicinity of the above values of parameters, the straight vortex line is slightly disturbed. It is obvious that, if the parameters change along the boundary (4.6), the vortex filament takes the form of a very elongated helicoidal filament, i.e. $h/a \gg 1$. When the parameters go into the existing region, the following two types of vortex filaments appear. One is

$$r = \epsilon\,\mathrm{cn}\,(2^{-\frac{1}{2}}\epsilon\xi \mid 2^{-\frac{1}{2}}), \quad \theta = \pm\xi + \theta_0, \quad z = \xi + z_0, \tag{4.10}$$

which is realized in the vicinity of the triple roots $\alpha = \beta = \gamma = 0$ (or $a = 0$, $h = 1$), viz., for $A = 2 + \frac{1}{4}\epsilon^4$ ($\epsilon \ll 1$) and $V = C = \pm 2$.

The other is

$$r = \frac{\epsilon}{\sigma}\left(h^2 + \frac{1}{h^2} + 2\cos\sigma\xi\right)^{\frac{1}{2}}, \quad z = \frac{h}{|h|}\xi + z_0, \tag{4.11a, b}$$

$$\theta = \pm\left\{\frac{h-1}{|h-1|}\tan^{-1}\left(\frac{\sin\sigma\xi}{h^2 + \cos\sigma\xi}\right) + \frac{\xi}{|h|}\right\} + \theta_0, \tag{4.11c}$$

where $\sigma = |h - 1/|h||$. This is realized when the parameters are near the double roots $\alpha = \beta = 0$ and $\gamma \neq 0$ (or $a = 0$ and $h \neq 1$), viz., for $A = 2h/|h| + \epsilon^2$, $V = \pm(h + 1/|h|)$ and $C = \pm(|h| + 1/h)$. In the particular case in which $h = -1$ ($C = V = 0$), it becomes

$$r = \epsilon\cos\xi, \quad \theta = \theta_0, \quad z = -\xi + z_0, \tag{4.12}$$

which represents a plane sinusoidal filament (Kelvin 1880).

4.5. *Solitary-wave-type filament*

If the modulus $k = 1$ (or $\beta = \gamma = 0$), the vortex filament takes the form

$$r = \sqrt{\alpha}\,\mathrm{sech}\,\tfrac{1}{2}\alpha^{\frac{1}{2}}\xi, \quad \theta = \pm\tfrac{1}{2}(4-\alpha)^{\frac{1}{2}}\xi + \theta_0, \quad z = \xi - \sqrt{\alpha}\tanh\tfrac{1}{2}\alpha^{\frac{1}{2}}\xi + z_0. \tag{4.13}$$

The parameters are given by $A = 2$, $C = V$ and $\alpha = 4 - V^2$ (> 0). The curvature κ and the torsion τ of this vortex filament are

$$\kappa(\xi) = \sqrt{\alpha}\,\mathrm{sech}\,\tfrac{1}{2}\alpha^{\frac{1}{2}}\xi, \quad \tau(\xi) = \frac{C}{2} = \text{constant} \tag{4.14}$$

(see (5.4) and (5.5) below), which is the solitary wave solution of NSE (5.1). The form (4.13) was first derived from (4.14) by Hasimoto (1972).

4.6. *Plane curves*

The plane curves are obtained most simply from (5.5) by demanding that the torsion of the curve should vanish. The following two cases are distinguished according as the curvature is constant or not.

When the curvature is constant along the filament, it becomes a circular ring discussed in § 4.2.

When the curvature is not constant, on the other hand, (5.5) yields $C = V = 0$. This means the vortex filament only rotates around the z-axis with uniform angular velocity Ω ($= 1$, see (3.2)), which is the situation examined by Hasimoto (1971). The solution for $|A| \leqslant 2$ is written as

$$r = 2k\,\mathrm{cn}\,(\xi \,|\, k), \quad \theta = \theta_0, \quad z = \xi - 2E(\xi \,|\, k) + z_0, \tag{4.15}$$

where $k = \frac{1}{2}(A+2)^{\frac{1}{2}}$. The modulus k roughly represents the amplitude of variation of the vortex filament. The curve (4.15) is known as Euler's elastica (Love 1927). Since the periods of r' and z' are $4K(k)$ and $2K(k)$ respectively, any closed plane curves without cross-points cannot exist. For very small values of k, (4.15) reduces to (4.12). As k increases, the filament becomes more bending, and at $k = 0{\cdot}8551$ ($A = 0{\cdot}925$) it starts to cross itself. Thereafter the solution is not acceptable since then LIE is not a good approximation. For $A \geqslant 2$, the solution is written as

$$r = \frac{2}{k}\,\mathrm{dn}\left(\frac{\xi}{k}\middle| k\right), \quad \theta = \theta_0, \quad z = \frac{A}{2}\xi - \frac{2}{k}E\left(\frac{\xi}{k}\middle| k\right) + z_0, \tag{4.16}$$

where $k = 2/(A+2)^{\frac{1}{2}}$. This solution is always crossed and is not acceptable.

5. Nonlinear Schrödinger equation

It is known that LIE yields NSE for the curvature and the torsion of the vortex filament:

$$(1/i)\,\partial\psi/\partial t = \psi'' + \tfrac{1}{2}\{|\psi|^2 + a(t)\}\,\psi, \tag{5.1}$$

where ψ is the complex variable,

$$\psi(s,t) = \kappa(s,t)\exp\left[i\int_0^s \tau(s',t)\,ds'\right] \tag{5.2}$$

and $a(t)$ is a real function of t (Hasimoto 1972). In this section we show that the solutions in the preceding sections correspond to the travelling-wave solutions of NSE.

The curvature and the torsion of the vortex filament are related to the differentials of $\mathbf{x}(s,t)$ by

$$\mathbf{x}' = \mathbf{t}, \quad \mathbf{x}'' = \kappa\mathbf{n}, \quad \mathbf{x}''' = -\kappa^2\mathbf{t} + \kappa'\,\mathbf{n} + \tau\kappa\mathbf{b}, \tag{5.3}$$

where $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are the unit vectors parallel to the tangent, the principal normal and the binormal of the vortex filament respectively. For the present solutions of steady motion, the curvature and the torsion depend only upon ξ (see (2.6)). By making use of the solutions in § 3, we can show that

$$\kappa^2(\xi) = R(\xi) + V^2 - C^2\,(\equiv \phi(\xi), \text{ say}), \tag{5.4}$$

$$\tau(\xi) = \frac{C}{2} - \frac{C^3 - CV^2 - AC + 2V}{2\kappa^2(\xi)}. \tag{5.5}$$

It follows from (3.9), (3.10) and (5.4) that the square of the curvature $\phi(\xi)$ satisfies the equation

$$\phi'^2+\phi^3-(2A+2V^2-3C^2)\phi^2+(A^2-4-4AC^2+2AV^2+4CV+3C^4-4C^2V^2+V^4)\phi$$
$$+(C^3-CV^2-AC+2V)^2=0. \quad (5.6)$$

If the curvature and the torsion are assumed to be functions of ξ only, (5.2) is written as

$$\psi(s,t)=\kappa(\xi)\exp\left[i\int_{-Ct}^{\xi}\tau(\xi')\,d\xi'\right], \quad (5.7)$$

which is the form of the travelling-wave solutions (see Scott, Chu & McLaughlin 1973). Introduction of this into (5.1) gives

$$\kappa''(\xi)=-\tfrac{1}{2}\kappa^3(\xi)+(\tau(\xi)-C)\,\tau(\xi)\,\kappa(\xi)-\tfrac{1}{4}B_1\kappa(\xi), \quad (5.8)$$

$$(2\tau(\xi)-C)\,\kappa'(\xi)+\tau'(\xi)\,\kappa(\xi)=0, \quad (5.9)$$

$$C\tau(-Ct)=\tfrac{1}{2}a(t)-\tfrac{1}{4}B_1, \quad (5.10)$$

where B_1 is a constant of integration. Equation (5.10) may be thought of as determining the function $a(t)$. Integration of (5.9) gives

$$\tau(\xi)=\frac{B_2}{2\kappa^2(\xi)}+\frac{C}{2}, \quad (5.11)$$

where B_2 is another constant of integration. Substituting (5.11) into (5.8) and integrating it, we find

$$\phi'^2+\phi^3+(C^2+B_1)\,\phi^2+B_3\phi+B_2^2=0, \quad (5.12)$$

where B_3 is the third constant of integration. Equation (5.12) is identical with (5.6) if we put

$$B_1=-2(A+V^2-C^2), \quad B_2=-C^3+CV^2+AC-2V, \quad (5.13a,b)$$

$$B_3=A^2-4-4AC^2+2AV^2+4CV+3C^4-4C^2V^2+V^4. \quad (5.13c)$$

Then (5.11) agrees with (5.5).

We have shown that our solutions (3.13), (3.15) and (3.16) correspond to the travelling wave solutions of NSE. It is interesting to note that the helicoidal filament, the straight line, the solitary-wave-type filament and the plane curves correspond to the nonlinear plane wave, the constant, the envelope solitary wave and the steady travelling-wave solutions of NSE respectively.

6. Discussion

We have investigated the motion of a vortex filament by making use of LIE. Recently it has been found that a one-dimensional classical spin system is described by

$$\partial\mathbf{S}(x,t)/\partial t=\mathbf{S}(x,t)\times\partial^2\mathbf{S}(x,t)/\partial x^2, \quad (6.1)$$

in the continuum limit, where $\mathbf{S}(x,t)$ is the unit spin vector and x a continuous variable (Lakshmanan *et al.* 1976; Lakshmanan 1977). This equation is made equivalent to LIE by identifying the unit spin vector $\mathbf{S}(x,t)$ with the unit tangent vector $\mathbf{t}$ of a vortex filament. Therefore our solutions also satisfy (6.1). In fact, Lakshmanan *et al.*

(1976) derived Betchov's intrinsic equation from (6.1) and found a few special solutions such as the spin waves, which correspond to our helicoidal filaments, and the solitary-wave-type solution.

It is important and interesting to investigate the stability of the solutions obtained in this paper. It is already known that a circular vortex ring is neutrally stable while a helicoidal filament is unstable to small disturbances (Betchov 1965; Kambe & Takao 1971). We are now investigating the stability in the general cases and the results will be reported in a future paper.

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Appendix

When $\Omega = 0$, we can solve (2.2)–(2.4) easily.

In the case $V = 0$, the inner product of $\mathbf{x}'$ and (2.4) together with (2.2) leads to $C = 0$. The outer product of $\mathbf{x}'$ and (2.4), on the other hand, by noting that $\mathbf{x}' . \mathbf{x}'' = 0$, gives $\mathbf{x}'' = 0$, which yields the straight line

$$\mathbf{x}(s,t) = \mathbf{a}s + \mathbf{b}, \quad |\mathbf{a}| = 1, \tag{A 1}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors independent of s and t. This may be included in the solutions for $V = C = 0$ and $r \equiv 0$ in § 3.

In the case $V \neq 0$, it is convenient to use a Cartesian co-ordinate system. After some algebra, we obtain, for $\mathbf{x} = (x, y, z)$,

$$x = \frac{(V^2 - C^2)^{\frac{1}{2}}}{V^2} \cos V(s - Ct - s_0) + x_0, \tag{A 2a}$$

$$y = \frac{(V^2 - C^2)^{\frac{1}{2}}}{V^2} \sin V(s - Ct - s_0) + y_0, \tag{A 2b}$$

$$z = \frac{C}{V}(s - Ct) + Vt + z_0, \tag{A 2c}$$

where x_0, y_0, z_0 and s_0 are constants and $V^2 \geqslant C^2$. This represents a helicoidal curve with the axis through (x_0, y_0, z_0) and parallel to the z-axis, and may be included in the solutions in § 4.2 by putting $a = (V^2 - C^2)^{\frac{1}{2}}/V^2$ and $h = C/V^2$.

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